1. Consider a Markov chain $\{X_n : \ n \geq 0\}$ with state space $S = \{0, 1, 2, 3\}$ and stationary transition probability matrix $P = ((p_{ij}))$ satisfying

$$\sum_{j=0}^{3} j \ p_{ij} = i \quad \text{for} \quad i = 0, 1, 2, 3.$$

Show that 0 and 3 must be absorbing states. [15]

2. Suppose that (X_i, Y_i), $i \geq 1$, are independent and identically distributed bivariate random vectors with $E(X_1) = \mu_x$, $E(Y_1) = \mu_y$, $Var(X_1) = \sigma_x^2$, $Var(Y_1) = \sigma_y^2$ and $Corr(X_1, Y_1) = \rho$. If X_1 and Y_1 are positive random variables, show that

$$Z_n = \sqrt{n}\left(\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} Y_i} - \frac{\mu_x}{\mu_y}\right)$$

converges in distribution to a normal random variable with mean 0 and variance $\frac{1}{\mu_y^4}(\mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2 - 2\rho\mu_x\mu_y\sigma_x\sigma_y)$. [15]

3. Consider a sequence of random variables $\{X_n : \ n \geq 1\}$, where $X_n \sim N(0, n^{-\alpha})$ for $n = 1, 2, \ldots$ and $\alpha > 0$ is fixed. Show that X_n converges almost surely to 0. [15]

4. Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with $P(X_i = 1) = p$ for $i = 1, 2, \ldots, n$, where $p \in [1/2, 1)$.

   (a) Is $T = n^{-1}\sum_{i=1}^{n} X_i$ a minimum variance unbiased estimator of p? Justify your answer.

   (b) Find an estimator T_0 such that $E(T - p)^2 > E(T_0 - p)^2$ for all $p \in [1/2, 1)$. [7+8]

5. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed normal variables with mean θ and variance 1. For the testing problem $H_0 : \theta = 0$ against $H_1 : \theta = 1$, find the critical region that minimizes $P(\text{Type I Error}) + 3\ P(\text{Type II Error})$. [15]

6. Consider the following data set

x	1	2	3	4	5	6
y	18.6	15.0	10.6	7.2	3.4	-1.1

Find the equation of the line $y = \alpha + \beta x$ that passes through the point $(5, 3)$ and minimizes $\sum_{i=1}^{6} |y_i - \alpha - \beta x_i|$, where (x_i, y_i) denotes the i-th $(i = 1, 2, \ldots, 6)$ observation. [15]

7. Consider the following block design D with 3 treatments denoted by 1, 2 and 3, assigned in 3 blocks of size 5.

$$D = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 3 & 1 \end{bmatrix}$$

(a) Identify a set of maximum number of linearly independent estimable functions of the treatment effects under the usual additive fixed effects two way ANOVA model.

(b) Find the best linear unbiased estimator of $\tau_1 - \tau_2$, where τ_i denotes the i-th $(i = 1, 2, 3)$ treatment effect. [3+12]

8. Let θ be the unknown proportion of tax evaders (denoted by attribute A) in a finite population of size N. In order to estimate θ, a statistician chose a sample of size n using simple random sampling without replacement. Because of the sensitive nature of A, a fair die was used. Two faces of the die were inscribed with A and the other four faces were inscribed with A^c. Each person in the sample was asked to roll the die till s/he got the face showing her/his true state and only report Z, the number of rolls required for it. These responses were denoted by $Z_1, Z_2, \ldots, Z_n$. Find

(a) an unbiased estimator T for θ,

(b) an unbiased estimator for $\text{Var}(T)$. [9+6]